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# Elastic rods, rigid bodies, quaternions and the last quadrature

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The Kirchhoff kinetic analogy relates the governing equations for the statics of elastic rods and the dynamics of rigid bodies. We discuss the analogy in light of several different Hamiltonian formulations, including a non-canonical description of rod equilibria. We focus on the last three quadratures that are required to reconstruct the rod centreline from the frame variables, which form the complete configuration space in the rigid body interpretation. In particular, we demonstrate that if the frame evolution is formulated as a canonical Hamiltonian system involving quaternions (or Euler parameters), then the last quadratures can all be computed explicitly in terms of algebraic relations involving invariants (or integrals) of the evolution, independent of whether or not the entire system is completely integrable.

## 1. Introduction

Long elastic rods provide a class of examples within solid mechanics that can exhibit localization effects both within equilibrium (e.g. Champneys & Thompson 1996*a*,*b*; van der Heijden *et al.* 1997) and dynamic (e.g. Dichmann *et al.* 1996; Coleman *et al.* 1993) theories. Within the special Cosserat description a rod is taken to be a framed curve, i.e. a centreline $\boldsymbol{r}$ along with an orthonormal frame $\{\boldsymbol{d}_i\}_{i=1}^3$, where for equilibria (or indeed for travelling wave solutions of dynamics) the functions $\boldsymbol{r}$ and $\{\boldsymbol{d}_i\}$ are each parametrized by a single variable $s$.

The three observations that follow in the main part of this article apply to quite general *hyperelastic* Cosserat rod models encompassing effects of shear and extension. (A rod is hyperelastic if there is an associated scalar strain energy function that generates all associated constitutive relations.) However, for the sake of brevity we shall here restrict ourselves to the case of an inextensible, unshearable rod, for which the pointwise constraint

$$\boldsymbol{r}' = \boldsymbol{d}_3 \tag{1.1}$$

is satisfied everywhere (more general cases can be found in Kehrbaum (1997)). Therefore, the parameter $s$ can be interpreted as arclength in any configuration and $\boldsymbol{d}_3$ coincides with the unit tangent to the centreline. In the absence of any external loads acting on the rod, such as gravity, the centreline $\boldsymbol{r}(s)$ does not explicitly appear in the potential energy of the rod, and as a consequence the force $\boldsymbol{n}(s)$ acting across

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each cross-section is in fact a constant $\boldsymbol{n}$ independent of $s$. Accordingly, the system of equations governing the equilibria of a rod can be decoupled into a part independent of $\boldsymbol{r}$ governing the frame $\{\boldsymbol{d}_i\}$ and once these equations have been solved (in some sense), the centreline $\boldsymbol{r}(s)$ can be recovered by carrying out the three quadratures inherent in the vector constraint (1.1). This is the observation at the heart of the *Kirchhoff kinetic analogy*, which, in its simplest form, states that for a linearly elastic rod, the reduced equations governing the rod frame $\{\boldsymbol{d}_i(s)\}$, parametrized by arclength $s$, are of a form identical with those governing the motion of a frame $\{\boldsymbol{d}_i(t)\}$, parametrized by time $t$, that is embedded in a heavy rigid body tumbling about a fixed point.

The Kirchhoff kinetic analogy is extremely useful for rod mechanics. Results from the vast literature concerning rigid body dynamics can be immediately translated to yield information for linearly elastic rods. In particular, the classic integrable case of the Lagrange top carries over to yield the integrable case of an isotropic, uniform rod, along with associated closed form solutions for the directors in terms of Euler angles (or other parametrizations of $SO(3)$). Recently, there has been renewed interest in tumbling rigid body dynamics as an example of a Hamiltonian system possessing a non-canonical Poisson bracket (cf. for example, Novikov 1982; Holmes & Marsden 1983) and, by invocation of the kinetic analogy, that non-canonical formulation can be exploited in the context of rods, as has been done by Mielke & Holmes (1988).

However, as is well known but sometimes ignored, the Kirchhoff kinetic analogy should not be pushed too far. For example, in the tumbling rigid body problem, the most natural side conditions involve an initial value problem, so that all the variables are known at $t = 0$, say. (Initial value problems combine well with closed form solutions known in completely integrable cases, because all the constants of integration arising in the associated quadratures are then known explicitly.) In contrast, many problems in rod mechanics involve two-point boundary value problems, so that unless the problem is statically determinate, the constant value for the force $\boldsymbol{n}$ is not known explicitly and must instead be determined as part of the solution satisfying prescribed boundary conditions. That issue has no simple analogy in rigid body dynamics. (It would correspond to motion in a gravity field whose direction and magnitude are unknown.)

The analogy is further strained by the fact that the centreline $\boldsymbol{r}(s)$ has no counterpart in the tumbling rigid body problem and the additional quadratures necessary to determine $\boldsymbol{r}(s)$ from (1.1) do not arise. Recently, it has been shown (from several rather different perspectives, e.g. Langer & Singer (1996); Shi & Hearst (1994); Ilyukhin (1979); Li (1986)) that the additional quadratures (1.1) can be carried out explicitly when the reduced problem corresponds to the integrable case of the Lagrange top. These closed form solutions provide much interesting information; for example, they imply that the centreline $\boldsymbol{r}(s) \in \mathbb{R}^3$ of all closed configurations of an isotropic, uniform rod lie on the surface of a torus of revolution. However, these closed form solutions again do not represent truly explicit solutions of many two-point boundary value problems for rods, because of the presence of undetermined constants. Tan & Witz (1995) discuss certain rod boundary value problems that can be solved explicitly, but these cases can in some sense be regarded as exceptional. In particular, while $\boldsymbol{r}(s)$ does not usually appear in the rod equilibrium equations, both $\boldsymbol{r}(0)$ and $\boldsymbol{r}(1)$ often do appear explicitly in the boundary conditions, and it is in such cases that the Kirchhoff kinetic analogy, the associated reduction to the tumbling rigid body problem and the last quadratures are particularly problematic.





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In this article we wish to make three observations. First we show that the unreduced rod problem can be written as a twelve-dimensional non-canonical Hamiltonian system in which the associated Poisson bracket is an extension of the rigid body Poisson bracket (cf. Novikov 1982; Holmes & Marsden 1983; Mielke & Holmes 1988). (To our knowledge this twelve-dimensional extended bracket has not been described previously, although it is closely related to a nine-dimensional bracket that was introduced and exploited within the context of satellite dynamics (see Wang *et al.* 1991, 1992).) The non-canonical Hamiltonian structure persists in the presence of various external potentials such as an external uniform gravity field. The existence and commutation relations of the integrals that arise in the extended system yield yet another perspective on the quadratures (1.1) and complete integrability of the rod problem.

Second, considerable interesting information is lost if the reduction inherent to the kinetic analogy is invoked too soon. In particular, we show that two of the three quadratures inherent in (1.1) can always be eliminated (or performed explicitly, depending on your point of view), independent of whether or not the problem is integrable. In particular, once a solution to the three degree-of-freedom tumbling rigid body problem has been found, a single additional scalar quadrature is required to reconstruct the centreline of the rod. The analysis we follow is a variant of that given in Ilyukhin (1979) (with a special case appearing in Landau & Lifshitz (1970)).

Third and finally, we observe that if the reduction inherent to the kinetic analogy is performed on a seven degree-of-freedom canonical Hamiltonian description of the rod equilibrium equations that exploits a quaternion (or Euler parameter) parametrization of the director frame (as introduced by Li & Maddocks (1997)), then all three of the quadratures (1.1) can be explicitly performed in terms of the variables appearing in the resulting four degree-of-freedom system. In particular, the additional degree of freedom introduced in the four parameter, singularity-free description of the locally three-dimensional group $SO(3)$ provides an additional integral that allows the last quadrature in (1.1) to be carried out explicitly. As a consequence, any two-point boundary value problem for a rod, including ones with boundary conditions explicitly involving both $\boldsymbol{r}(0)$ and $\boldsymbol{r}(1)$, can be rewritten as a two-point boundary value problem for the reduced four degree-of-freedom system that describes the frame evolution in terms of quaternions.

The paper is structured as follows. Section 2 contains a brief synopsis of necessary background material, namely a description of the equilibrium theory of Cosserat rods in § 2 *a*, a description of the equations of a tumbling rigid body and the Kirchhoff kinetic analogy in § 2 *b* and certain relevant issues concerning Hamiltonian systems in § 2 *c*. Then § 3 deals with different Hamiltonian formulations of the equations of tumbling rigid body dynamics and the equilibria of elastic rods. Finally in § 4 we discuss how to reduce the quadratures (1.1) to a single, scalar (*last*) *quadrature*; the latter can be resolved by exploiting the canonical quaternion Hamiltonian description of elastic rods.

## 2. Background material

### (*a*) *Cosserat theory of elastic rods*

In this section we review the director theory of elastic, inextensible and unshearable (Cosserat) rods (cf. Antman 1995, ch. VIII, IX). The configuration of a rod is specified by a smooth curve $\boldsymbol{r}(s)$, that can be interpreted as the centreline of the rod, and an





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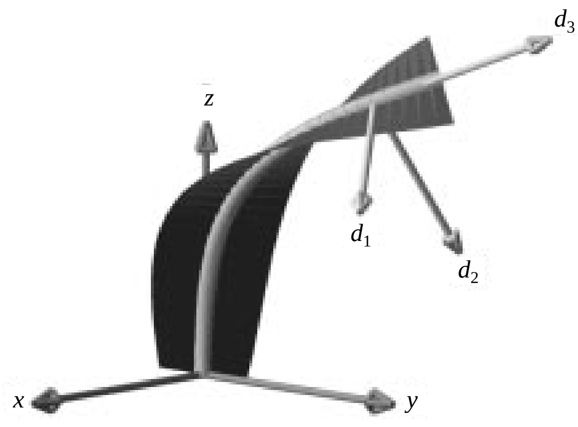


Figure 1. A typical configuration of a rod. The tube represents the centreline and the director $\boldsymbol{d}_1$ traces the ribbon, representing twisting.

orthonormal frame $\{\boldsymbol{d}_i\}_{i=1}^3$, called the *directors*, that serves to locate the orientation of a cross-section of the rod (cf. figure 1). We assume that the rod is effectively unshearable and inextensible. In mathematical terms, inextensibility is expressed as the condition $|\boldsymbol{r}'| = 1$ and unshearability is $\boldsymbol{r}' \cdot \boldsymbol{d}_1 = \boldsymbol{r}' \cdot \boldsymbol{d}_2 = 0$. Consequently, we may write $\boldsymbol{r}' = \boldsymbol{d}_3$, so that the parameter $s$ can be interpreted as non-dimensionalized arclength in any configuration of the rod. It will be natural to express components of vectors with respect to the director frame; for any vector $\boldsymbol{p}$, the triple of components $p_i = \boldsymbol{p} \cdot \boldsymbol{d}_i$ will be denoted with the sans-serif symbol $\mathsf{p}$. The triple $\mathsf{p}$ will be referred to as the body components of $\boldsymbol{p}$.

Orthonormality of the directors $\{\boldsymbol{d}_i\}$ implies the existence of a *strain vector* $\boldsymbol{u}$ satisfying

$$\boldsymbol{d}_i' = \boldsymbol{u} \times \boldsymbol{d}_i, \quad i = 1, 2, 3. \tag{2.1}$$

The components $u_i = \boldsymbol{u} \cdot \boldsymbol{d}_i$ with respect to the frame $\{\boldsymbol{d}_i\}$ are the bending $(i = 1, 2)$ and twisting $(i = 3)$ strains.

In rod mechanics, the forces and moments acting across a material cross-section are averaged to yield a net force $\boldsymbol{n}(s)$ and a net moment $\boldsymbol{m}(s)$. Then balance of moments and forces imply the coordinate-free equilibrium equations

$$\boldsymbol{m}' + \boldsymbol{r}' \times \boldsymbol{n} = \boldsymbol{0}, \quad \boldsymbol{n}' = \boldsymbol{0}. \tag{2.2}$$

The components $m_i = \boldsymbol{m} \cdot \boldsymbol{d}_i$ are the bending and twisting moments in the rod. The moments $m_i$ must be related to the strains $u_i$ via suitable constitutive relations. Here we make the assumption of hyperelasticity: for $\mathsf{v} = (v_1, v_2, v_3)$, there exists a convex, coercive strain energy density function $W(\mathsf{v}, s)$ such that $W_{\mathsf{v}}(\boldsymbol{0}, s) = \boldsymbol{0}$. The function $W$ is *coercive*, if

$$\lim_{|\mathsf{v}| \to \infty} \frac{W(\mathsf{v}, s)}{|\mathsf{v}|} = \infty$$

(cf. Antman 1995, ch. VIII). The constitutive relations are taken to be of the form

$$\mathsf{m} = W_{\mathsf{v}}(\mathsf{u} - \hat{\mathsf{u}}, s), \tag{2.3}$$

where $\hat{\mathsf{u}}(s) = (\hat{u}_1(s), \hat{u}_2(s), \hat{u}_3(s))$ are the components of strain in the minimum energy unstressed configuration. A standard approximation in rod theory is to assume the rod to be linearly elastic (i.e. the strain energy is a quadratic function of the strains); in particular, one often adopts the diagonal quadratic form

$$W(\mathsf{v}, s) = \frac{1}{2}\sum_{i=1}^{3} K_i(s) v_i^2. \tag{2.4}$$

This example of the Cosserat theory is often called the Kirchhoff rod.

We will call the constitutive relation *isotropic* if the strain energy density function $W(\mathsf{v}, s)$ is invariant under rotations about the $\boldsymbol{d}_3$-axis; equivalently,

$$\frac{\partial}{\partial \alpha} W(\kappa \cos\alpha, \kappa \sin\alpha, v_3, s) = 0,$$

for all $0 \leqslant \alpha < 2\pi$, $\kappa > 0$ and $v_3 \in \mathbb{R}$ (cf. Maddocks 1984). The constitutive relation (2.4) is isotropic if $K_1(s) \equiv K_2(s)$. We will further describe the *rod* as isotropic if it has no preferred material direction for bending; provided that the constitutive relation is isotropic, the rod is isotropic if, in addition, $\hat{u}_1(s) \equiv \hat{u}_2(s) \equiv 0$. The constitutive relation is *uniform* if the strain energy function $W$ does not explicitly depend on arclength $s$. The rod is uniform if, in addition, the components of unstressed strain $\hat{\mathsf{u}}$ are constant.

### (*b*) *Kirchhoff's kinetic analogy and the dynamics of tumbling rigid bodies*

According to Love (1944), the classic form of the Kirchhoff kinetic analogy is based on a rod theory with linear constitutive relations of the form

$$\mathsf{m} = \mathsf{K}(s)(\mathsf{u} - \hat{\mathsf{u}}(s)), \tag{2.5}$$

where $\mathsf{K}$ is a positive definite matrix. If the matrix $\mathsf{K}$ is constant and $\hat{\mathsf{u}} \equiv \mathbf{0}$, equations (2.2) are equivalent to the classic equations of motion of a heavy rigid body in a uniform gravitational field tumbling about a fixed point; in this case, the frame $\{\boldsymbol{d}_i\}$ is fixed in the body (and referred to as the body frame) in such a way that the centre of mass lies on the $\boldsymbol{d}_3$-axis, the matrix $\mathsf{K}$ is the inertia tensor of the rigid body about the fixed point referred to the $\{\boldsymbol{d}_i\}$ frame and equations (2.2) are typically rewritten as

$$\boldsymbol{m}' + \boldsymbol{d}_3 \times \boldsymbol{n} = \mathbf{0}, \quad \boldsymbol{n}' = \mathbf{0}, \tag{2.6}$$

with the independent parameter $s$ now interpreted as time (and usually written $t$). We shall refer to this problem as the tumbling rigid body. It is also often assumed that the body frame coincides with the principal axes of inertia, so that $\mathsf{K}$ is a diagonal matrix with entries $K_1 \leqslant K_2 \leqslant K_3$, which is a restriction on the possible locations of the centre of mass. Nevertheless, in this special case, the equations describing the dynamics of a tumbling rigid body are identical in form to those describing the equilibria of a Kirchhoff rod with diagonal strain energy density (2.4).

Two important special cases in which the problem is completely integrable (in the classic sense) have been studied extensively. A tumbling rigid body that is free to move with no external forces acting on it (or equivalently the fixed point is at the centre of mass) is often described as the free rigid body. In the context of rods the free rigid body corresponds to special problems where the end-loadings are pure moments, so that $\boldsymbol{n} = \mathbf{0}$. The second case is that of a Lagrange top, which is a tumbling rigid body with a diagonalized inertia tensor in which $K_1$ and $K_2$ are equal. The Lagrange top corresponds to an isotropic, uniform Kirchhoff rod.

A less widely known fact is that the Kirchhoff kinetic analogy persists for non-zero $\hat{\boldsymbol{u}}$. For example, Wittenburg (1977) considers a gyrostat comprising a carrier rigid body in which is mounted a spinning sub-body (often called a momentum wheel). The function $\hat{\boldsymbol{u}}$ can then be interpreted as the angular velocity of the momentum wheel relative to the carrier and the analogy applies if one assumes that these relative velocities are prescribed functions of time. (The analogy can also be made for more general systems; for example, for gyrostats with multiple momentum wheels.)

### (c) *Hamiltonian systems*

We will require some basic concepts and terminology from finite-dimensional Hamiltonian systems (cf. for example, Olver 1993, ch. 6). Let $\boldsymbol{X}$ be a smooth, $m$-dimensional manifold (often $\boldsymbol{X} = \mathbb{R}^{2n}$). A Poisson bracket is an operator that assigns a smooth real-valued function $\{F, G\}$ to each pair of real-valued, smooth functions $F$ and $G$ defined on $\boldsymbol{X}$. The operation is bilinear, skew-symmetric and satisfies the Jacobi identity and the Leibniz rule. In local coordinates $\boldsymbol{z} = (z_1, \ldots, z_m)$, any Poisson bracket can be written

$$\{F, G\}(\boldsymbol{z}) = \nabla F(\boldsymbol{z}) \cdot \mathcal{J}(\boldsymbol{z}) \nabla G(\boldsymbol{z}),$$

where $\mathcal{J}(\boldsymbol{z}) = (\mathcal{J}_{ij}(\boldsymbol{z}))$ is a skew-symmetric *structure matrix* defined by $\mathcal{J}_{ij}(\boldsymbol{z}) = \{z_i, z_j\}$, $1 \leqslant i, j, \leqslant m$, and $\nabla$ denotes the gradient with respect to the phase variables $\boldsymbol{z}$. Given a Hamiltonian function $\mathcal{H} : \boldsymbol{X} \times \boldsymbol{R} \to \mathbb{R}$, a Hamiltonian system is the $m$-dimensional system of ordinary differential equations

$$\boldsymbol{z}' = \mathcal{J}(\boldsymbol{z}) \nabla \mathcal{H}(\boldsymbol{z}, t), \tag{2.7}$$

so that for any solution $\boldsymbol{z}(t)$, and any smooth function $F(\boldsymbol{z})$,

$$\frac{\mathrm{d}}{\mathrm{d}t} F(\boldsymbol{z}) = \{F, H\}(\boldsymbol{z}).$$

The coordinates $\boldsymbol{z}$ on an even-dimensional manifold $\boldsymbol{X} = \mathbb{R}^{2n}$ will be called canonical if the associated structure matrix is the standard constant matrix

$$\boldsymbol{J}_{2n} = \begin{pmatrix} \boldsymbol{0} & \boldsymbol{I} \\ -\boldsymbol{I} & \boldsymbol{0} \end{pmatrix}, \tag{2.8}$$

where $\boldsymbol{I}$ and $\boldsymbol{0}$ are the identity and null matrices of dimension $n$.

A function $I : \boldsymbol{X} \to \mathbb{R}$ is an integral of (2.7) if it is constant along trajectories. Equivalently, $I$ is an integral if

$$\{I, \mathcal{H}\}(\boldsymbol{z}) = \nabla I(\boldsymbol{z}) \cdot \mathcal{J}(\boldsymbol{z}) \nabla \mathcal{H}(\boldsymbol{z}, t) = 0.$$

In particular, there exist integrals called Casimirs that exist for any Hamiltonian. The function $C : \boldsymbol{X} \to \mathbb{R}$ is a Casimir if $\{C, F\} = 0$ for all smooth functions $F$. Equivalently, $C$ is a Casimir if $\nabla C$ is in the null-space of $\mathcal{J}$ (cf. Olver 1993). In a canonical formulation there are no Casimirs since the operator $\boldsymbol{J}_{2n}$ is non-singular.

The Hamiltonian system (2.7) is called completely integrable if its solution can be reduced to quadratures, so that (at least in principle) a solution to the initial value problem can be found in closed form. Two integrals $I_1$ and $I_2$ are said to be in involution (or to commute) if $I_1$ is an integral with respect to the flow generated by $I_2$, i.e.

$$\{I_1, I_2\}(\boldsymbol{z}) = \nabla I_1(\boldsymbol{z}) \cdot \mathcal{J}(\boldsymbol{z}) \nabla I_2(\boldsymbol{z}) = 0.$$

Suppose the Hamiltonian system (2.7) possesses $k$ Casimirs and an additional $l$ integrals in involution whose gradients are linearly independent. Then the system is completely integrable if

$$2l + k = m.$$

Notice that there may well be more than $\frac{1}{2}(m-k)$ integrals with linearly independent gradients, without there being a sufficient number of commuting integrals for the system to be completely integrable. This circumstance arises in the Hamiltonian systems governing rod equilibria that we describe below.

## 3. Hamiltonian formulations

In this section we derive several Hamiltonian formulations governing the equilibria of elastic rods and the motion of a tumbling rigid body. In § 3 *a* we review a non-canonical Hamiltonian formulation for the dynamics of a tumbling rigid body (cf. Novikov 1982; Holmes & Marsden 1983). This Hamiltonian rigid body formulation has been used by Mielke & Holmes (1988) in the context of rod mechanics to determine the orientation of the director frame, and they then recover the spatial components of the variable $\boldsymbol{r}$ by solving the additional differential equations (1.1) and (2.1), but from a non-Hamiltonian perspective. In § 3 *b* we present an extended non-canonical Hamiltonian formulation that includes the body components of both the centreline $\boldsymbol{r}$ and an additional fixed vector $\boldsymbol{i}$ in the phase space. The integrals of this extended Hamiltonian system provide a different perspective on the final quadratures (1.1). The spatial components of the centreline $\boldsymbol{r}$ can be obtained by simple algebraic manipulations from the solution in the body coordinate phase space. A more classical approach is to obtain a canonical Hamiltonian formulation by combining the spatial components of $\boldsymbol{r}$ with some parametrization of the rotation group $SO(3)$. In § 3 *c* we describe a seven degree-of-freedom canonical Hamiltonian system that arises in this way when a parametrization of $SO(3)$ in terms of quaternions is adopted. Finally, in § 3 *d* we discuss the various manifestations of integrals, commutativity properties and completely integrable cases that arise in the different Hamiltonian formulations.

### (*a*) *Non-canonical formulation for the dynamics of a tumbling rigid body*

In this formulation we rewrite the balance equations (2.6) in body coordinates

$$\mathsf{m}' = \mathsf{m} \times \mathsf{u} + \mathsf{n} \times \mathsf{e}_3, \tag{3.1}$$

$$\mathsf{n}' = \mathsf{n} \times \mathsf{u}, \tag{3.2}$$

where $\mathsf{e}_3 = (0,0,1)^{\mathrm{T}}$ and $\mathsf{u}$ and $\mathsf{m}$ are related via (2.5). The Hamiltonian structure of this system has been investigated by Novikov (1982) and Holmes & Marsden (1983), for example. Specifically, a Poisson bracket can be defined in local coordinates by the structure matrix

$$\boldsymbol{J}(\mathsf{m}, \mathsf{n}) = \begin{pmatrix} \mathsf{m}\times & \mathsf{n}\times \\ \mathsf{n}\times & \mathbf{0} \end{pmatrix}, \tag{3.3}$$

where for any triple $\mathsf{p} = (p_1, p_2, p_3)$ we use the notation $\mathsf{p}\times$ to denote the matrix

$$\mathsf{p}\times = \begin{pmatrix} 0 & -p_3 & p_2 \\ p_3 & 0 & -p_1 \\ -p_2 & p_1 & 0 \end{pmatrix},$$

corresponding to the vector product. Then equations (3.1) and (3.2) are equivalent to the Hamiltonian system

$$\begin{pmatrix} m \\ n \end{pmatrix}' = \boldsymbol{J}(m, n)\nabla H(m, n, s), \tag{3.4}$$

with Hamiltonian

$$H(m, n, s) = \tfrac{1}{2} m \cdot K^{-1} m + \hat{u}(s) \cdot m + n \cdot e_3.$$

Equations (3.4) are the well-known equations of Euler and Poisson for the motion of a tumbling rigid body.

### (b) *Non-canonical formulation for equilibria of rods*

We now rewrite the equations governing rod equilibria entirely in terms of body variables to extend the formulation of the previous section to a twelve-dimensional non-canonical Hamiltonian system. Assume $\boldsymbol{n} \neq \boldsymbol{0}$, let $\boldsymbol{k} = \boldsymbol{n}/|\boldsymbol{n}|$, $\boldsymbol{i}$ be a fixed unit vector and define $\boldsymbol{j} = \boldsymbol{k} \times \boldsymbol{i}$. Then $\{\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\}$ define a basis of $\boldsymbol{R}^3$, provided that $\boldsymbol{i}$ and $\boldsymbol{n}$ are non-collinear.

We rewrite the differential equations (1.1) and (2.2) in terms of body coordinates to obtain

$$r' = r \times u + e_3, \tag{3.5}$$

$$m' = m \times u + n \times e_3, \tag{3.6}$$

$$n' = n \times u. \tag{3.7}$$

To complete the description we also rewrite the trivial equation $\boldsymbol{i}' = \boldsymbol{0}$ in body coordinates, i.e.

$$i' = i \times u, \tag{3.8}$$

to obtain an additional Poisson equation. The constitutive relation (2.3) relates $u$ and $m$, which can be inverted to yield

$$u = \frac{\partial W^*(m, s)}{\partial m} + \hat{u},$$

where $W^*(m, s)$ is the Legendre transform of $W(u, s)$, i.e.

$$W^*(m, s) = \sup_{v \in \mathbb{R}^3} \{m \cdot v - W(v, s)\} \tag{3.9}$$

(cf. Li & Maddocks 1997; Kehrbaum & Maddocks 1997). Equations (3.5)–(3.8) can then be seen to possess a Hamiltonian structure. Define the Hamiltonian

$$H(i, r, m, n, s) = W^*(m, s) + m \cdot \hat{u} + n \cdot e_3. \tag{3.10}$$

Then Hamilton's equations are of the form

$$\begin{pmatrix} i \\ r \\ m \\ n \end{pmatrix}' = \boldsymbol{J}(i, r, m, n)\nabla H(i, r, m, n, s), \tag{3.11}$$

where the skew-symmetric operator $\boldsymbol{J}$ is defined by

$$\boldsymbol{J}(i,r,m,n) = \begin{pmatrix} \mathbf{0} & \mathbf{0} & i\times & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & r\times & \boldsymbol{I} \\ i\times & r\times & m\times & n\times \\ \mathbf{0} & -\boldsymbol{I} & n\times & \mathbf{0} \end{pmatrix}. \tag{3.12}$$

In order that $\boldsymbol{J}$ actually generate a Poisson bracket, the Jacobi identity must be verified (cf. Olver 1993, proposition 6.8). The necessary calculations are computationally tedious, but have been directly checked using a symbolic manipulation package (Mathematica).

For the non-canonical formulation to be meaningful it is important that the fixed frame components of all variables be determined from the phase variables $i$, $r$, $m$ and $n$. We now show that this is indeed the case. For convenience we assume that the fixed frame $\{\boldsymbol{i},\boldsymbol{j},\boldsymbol{k}\}$ is orthonormal. The rotation matrix can then be constructed in terms of the nine direction cosines relating the fixed frame $\{\boldsymbol{i},\boldsymbol{j},\boldsymbol{k}\}$ and the director frame $\{\boldsymbol{d}_i\}_{i=1}^3$. We write the direction cosines in the form

$$\boldsymbol{R} = \begin{pmatrix} \boldsymbol{d}_1\cdot\boldsymbol{i} & \boldsymbol{d}_1\cdot\boldsymbol{j} & \boldsymbol{d}_1\cdot\boldsymbol{k} \\ \boldsymbol{d}_2\cdot\boldsymbol{i} & \boldsymbol{d}_2\cdot\boldsymbol{j} & \boldsymbol{d}_2\cdot\boldsymbol{k} \\ \boldsymbol{d}_3\cdot\boldsymbol{i} & \boldsymbol{d}_3\cdot\boldsymbol{j} & \boldsymbol{d}_3\cdot\boldsymbol{k} \end{pmatrix} = (i,j,k) = \left(i, \frac{n}{|n|}\times i, \frac{n}{|n|}\right).$$

Then the triple $p$ of body components of any vector is related to the triple $\underline{\boldsymbol{p}}$ of components in the fixed frame $\{\boldsymbol{i},\boldsymbol{j},\boldsymbol{k}\}$ through the relation $p = \boldsymbol{R}\underline{\boldsymbol{p}}$, or equivalently $\underline{\boldsymbol{p}} = \boldsymbol{R}^{\mathrm{T}}p$.

It is noteworthy that the Hamiltonian structure defined by (3.12) persists under the presence of a gravitational field acting in the direction $\boldsymbol{i}$, say. For such a heavy elastic rod the equilibrium equations are given by (3.11), with $H$ replaced by the Hamiltonian

$$\tilde{H}(i,r,m,n,s) = W^*(m,s) + \hat{u}\cdot m + n\cdot e_3 + \gamma r\cdot i,$$

with $\gamma$ an appropriate parameter.

### (*c*) *Canonical formulations for rod equilibria and dynamics of tumbling rigid bodies*

To obtain a canonical Hamiltonian description of the rod equilibrium equations that directly includes the spatial components of the variables $\boldsymbol{r}$ and $\boldsymbol{n}$ as phase variables, we introduce a parametrization for the orthonormal directors $\{\boldsymbol{d}_i\}$, i.e. a parametrization of the group $SO(3)$ of $3\times3$ proper rotation matrices. The classic choice of parametrization is Euler angles (cf. for example, Antman 1995, ch. VIII). Any appropriate choice of three angles provides a local parametrization of $SO(3)$, but a polar singularity of some form always arises, which can give rise to awkward analytical and numerical difficulties. In contrast we shall adopt the four parameter description of $SO(3)$ in terms of quaternions that provides a singularity-free, double covering.

In § 3 *c* (i) we discuss the derivation of a seven degree-of-freedom Hamiltonian system involving quaternions. For completeness, in § 3 *c* (ii) we state an analogous six degree-of-freedom system in terms of Euler angles that is valid away from the polar singularity. Then in § 3 *c* (iii) we show how a reduction leads to lower-dimensional systems describing the dynamics of tumbling rigid bodies.

(i) *Quaternion formulation*

We now outline a canonical Hamiltonian formulation of the rod equilibrium equations in terms of quaternions due to Li & Maddocks (1997), which should be consulted for additional details. The derivation of the Hamiltonian system is not entirely standard because the adoption of a four parameter description of $SO(3)$ means that the appropriate Lagrangian is not a strictly convex function of the generalized velocities.

A quaternion is defined by a vector $\boldsymbol{q} = (q_1, q_2, q_3, q_4) \in \mathbb{R}^4$ satisfying the norm constraint

$$|\boldsymbol{q}| = 1. \tag{3.13}$$

An orthonormal set of directors $\{\boldsymbol{d}_i\}$ can be explicitly written in terms of the quaternion $\boldsymbol{q}$, e.g.

$$\boldsymbol{d}_3 = \begin{pmatrix} 2(q_1q_3 + q_2q_4) \\ 2(-q_1q_4 + q_2q_3) \\ -q_1^2 - q_2^2 + q_3^2 + q_4^2 \end{pmatrix}.$$

With the notation (see, for example, Dichmann 1994)

$$\boldsymbol{B}_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \quad \boldsymbol{B}_2 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad \boldsymbol{B}_3 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix},$$

the strains $u_i$ can be expressed in terms of the quaternion $\boldsymbol{q}$ and its derivative

$$u_i = \frac{2\boldsymbol{B}_i\boldsymbol{q} \cdot \boldsymbol{q}'}{|\boldsymbol{q}|^2}.$$

The rod equilibria are then the constrained critical points of the energy functional

$$\int_0^1 W(u(\boldsymbol{q}, \boldsymbol{q}') - \hat{u}, s)\, \mathrm{d}s,$$

subject to the constraints of inextensibility and unshearability (1.1), and the quaternion norm condition written in the form $\boldsymbol{q}(s) \cdot \boldsymbol{q}'(s) = 0$. Equivalently, one can introduce Lagrange multipliers $\boldsymbol{\lambda}(s) \in \mathbb{R}^3$ and $\nu \in \mathbb{R}$ such that the equilibria are unconstrained critical points of the Lagrangian

$$\mathcal{L}(\boldsymbol{r}, \boldsymbol{r}', \boldsymbol{q}, \boldsymbol{q}', s) = W(u(\boldsymbol{q}, \boldsymbol{q}') - \hat{u}, s) + \boldsymbol{\lambda} \cdot (\boldsymbol{r}' - \boldsymbol{d}_3) + \nu \boldsymbol{q} \cdot \boldsymbol{q}'.$$

The conjugate variables are introduced through the definitions

$$\boldsymbol{n} = \frac{\partial \mathcal{L}}{\partial \boldsymbol{r}'} = \boldsymbol{\lambda}, \quad \boldsymbol{\mu} = \frac{\partial \mathcal{L}}{\partial \boldsymbol{q}'} = \frac{2}{|\boldsymbol{q}|^2} \sum_i W_{u_i} \boldsymbol{B}_i \boldsymbol{q} + \nu \boldsymbol{q}.$$

The conjugate variable $\boldsymbol{n}$ is merely the net force introduced in § 2 *a*. A complete physical interpretation of the conjugate variable $\boldsymbol{\mu}$ is less obvious, but the moments $\boldsymbol{m} = (m_1, m_2, m_3)$ are given in terms of $\boldsymbol{q}$ and $\boldsymbol{\mu}$ by the expressions

$$m_i = \tfrac{1}{2} \boldsymbol{B}_i \boldsymbol{q} \cdot \boldsymbol{\mu}. \tag{3.14}$$

It can then be verified that the rod equilibrium conditions are equivalent to the seven





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degree-of-freedom Hamiltonian system

$$\begin{pmatrix} \boldsymbol{r} \\ \boldsymbol{q} \\ \boldsymbol{n} \\ \boldsymbol{\mu} \end{pmatrix}' = \boldsymbol{J}_{14} \nabla H^{\boldsymbol{q}}(\boldsymbol{r}, \boldsymbol{q}, \boldsymbol{n}, \boldsymbol{\mu}, s), \tag{3.15}$$

in which the operator $\boldsymbol{J}_{14}$ corresponds to the canonical Poisson bracket defined in (2.8) and the Hamiltonian is merely (3.10) expressed in terms of the new variables

$$H^{\boldsymbol{q}}(\boldsymbol{r}, \boldsymbol{q}, \boldsymbol{n}, \boldsymbol{\mu}, s) = W^*(m, s) + m \cdot \hat{u} + \boldsymbol{n} \cdot \boldsymbol{d}_3,$$

using (3.14).

(ii) *Euler angle formulation*

With a specific sequence of Euler angles $\boldsymbol{\phi} = (\phi, \theta, \psi)$ (we adopt here the particular sequence of Euler angles that can be found, for example, in Dichmann (1994) or Maddocks (1984)), one can parametrize the directors as e.g.

$$\boldsymbol{d}_3 = \begin{pmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{pmatrix}.$$

As above, one constructs a constrained variational principle and introduces conjugate variables $\boldsymbol{n}$ and $\boldsymbol{p}_\phi = (p_\phi, p_\theta, p_\psi)$ to obtain a six degree-of-freedom Hamiltonian system

$$\begin{pmatrix} \boldsymbol{r} \\ \boldsymbol{\phi} \\ \boldsymbol{n} \\ \boldsymbol{p}_\phi \end{pmatrix}' = \boldsymbol{J}_{12} \nabla H^{\phi}(\boldsymbol{r}, \boldsymbol{\phi}, \boldsymbol{n}, \boldsymbol{p}_\phi, s),$$

where the Hamiltonian is (3.10) rewritten as

$$H^{\phi}(\boldsymbol{r}, \boldsymbol{\phi}, \boldsymbol{n}, \boldsymbol{p}_\phi, s) = W^*(m, s) + \hat{u} \cdot m + \boldsymbol{n} \cdot \boldsymbol{d}_3,$$

in which

$$m_1 = -p_\phi \cos\psi \csc\theta + p_\theta \sin\psi + p_\psi \cos\psi \cot\theta,$$
$$m_2 = p_\phi \csc\theta \sin\psi + p_\theta \cos\psi - p_\psi \cot\theta \sin\psi, \quad m_3 = p_\psi.$$

(Compare, for example, Holmes & Marsden 1983.) Notice that the polar singularity is manifested in the appearance of $\csc\theta$ and $\cot\theta$ terms.

(iii) *Formulations for tumbling rigid body dynamics*

In both of the above canonical formulations the centreline $\boldsymbol{r}$ represents three ignorable (or cyclic) variables, i.e. the variable $\boldsymbol{r}$ does not appear explicitly in the Hamiltonian. Accordingly, the conjugate variables $\boldsymbol{n}$ automatically provide three commuting integrals that may therefore be used to reduce the dimension of the Hamiltonian system by six. Thus, for prescribed values of $\boldsymbol{n}$, the reduction to the frame variables inherent to the Kirchhoff kinetic analogy is entirely standard from the Hamiltonian perspective. Notice, however, that with the quaternion representation the reduced system has four degrees of freedom, whereas a parametrization in terms of Euler angles leads to a reduced system with three degrees of freedom.

### (*d*) *Integrals and complete integrability*

We now describe the integrals that arise in the various Hamiltonian formulations for the elastic rod and the tumbling rigid body, and discuss associated completely integrable systems.

Considering the balance laws (2.2) it is apparent that the functions

$$\boldsymbol{m} + \boldsymbol{r} \times \boldsymbol{n} \tag{3.16}$$

and

$$\boldsymbol{n} \tag{3.17}$$

must be integrals of any of the Hamiltonian systems describing the equilibria of elastic rods. We will show how these (and additional) integrals arise in each of the formulations.

#### (i) *Integrals for the non-canonical formulation*

Consider the non-canonical Hamiltonian system (3.11). We need to distinguish trivial integrals that arise due to degeneracies of the operator $\boldsymbol{J}$ (i.e. Casimirs) and integrals that depend on the specific Hamiltonian under study. In our case, the operator (3.12) has a two-dimensional null-space, spanned by the vectors

$$\nabla C_1 = \begin{pmatrix} \mathsf{m} + \mathsf{r} \times \mathsf{n} \\ \mathsf{n} \times \mathsf{i} \\ \mathsf{i} \\ \mathsf{i} \times \mathsf{r} \end{pmatrix}, \quad \nabla C_2 = \begin{pmatrix} \mathsf{i} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{pmatrix}.$$

The associated distinguished integrals are $C_1 = \mathsf{i} \cdot (\mathsf{m} + \mathsf{r} \times \mathsf{n})$ and $C_2 = \frac{1}{2}|\mathsf{i}|^2$; the integral $C_2$ arises since $\boldsymbol{i}$ is a fixed vector, while the integral $C_1$ is the component of (3.16) with respect to $\boldsymbol{i}$ written in body coordinates.

Consider a general Hamiltonian $\mathcal{H}(\mathsf{m}, \mathsf{n}, s)$, dependent only on the variables $\mathsf{m}$ and $\mathsf{n}$ (and $s$), but not $\mathsf{r}$ and $\mathsf{i}$. Then it is easy to verify that the components of $\boldsymbol{m}+\boldsymbol{r}\times\boldsymbol{n}$ with respect to $\boldsymbol{n}$ and $\boldsymbol{j}$ are integrals with respect to $\mathcal{H}$; in body coordinates these integrals are $\mathsf{m} \cdot \mathsf{n}$ and $(\mathsf{n} \times \mathsf{i}) \cdot (\mathsf{m} + \mathsf{r} \times \mathsf{n})$. Furthermore, the components of the constant force vector $\boldsymbol{n}$ in the basis $\{\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\}$ are trivially integrals of $\mathcal{H}$; in body coordinates $|\mathsf{n}|^2$ and $\mathsf{n} \cdot \mathsf{i}$ are the only independent integrals. In the absence of further symmetry properties of the constitutive function, and therefore of the Hamiltonian, the six quantities (3.16) and (3.17) constitute all known integrals. However, if the rod is isotropic, the twisting moment $m_3$ is also an integral, and if the rod is uniform, the Hamiltonian becomes autonomous and so itself provides an additional integral.

#### (ii) *Integrals for the heavy rod equations*

For the Hamiltonian description of a heavy elastic rod the two Casimir integrals persist, since they form the null-space of (3.12). While all remaining integrals arising for system (3.11) are individually destroyed, a linear combination of them persists, namely $|\mathsf{n} \times \mathsf{i}|^2 = |\mathsf{j}|^2$; and, as above, the integrals $m_3$ and $H$ arise if the rod is isotropic and uniform, respectively.

(iii) *Integrals for the quaternion formulation*

The integrals (3.16) and (3.17) are independent of the particular variables adopted; they must persist in the canonical Hamiltonian formulation (3.15) written in terms of quaternions (or indeed Euler angles) and merely need to be re-expressed in terms of the quaternion $\boldsymbol{q}$ and its conjugate momentum $\boldsymbol{\mu}$.

The point of interest is that the parametrization in terms of quaternions gives rise to *additional* integrals (see Li & Maddocks (1997) for more details and a discussion of the associated symmetries). First, the quaternion norm constraint (3.13) is manifested as the integral

$$\boldsymbol{q}\cdot\boldsymbol{q}, \tag{3.18}$$

associated with the gauge freedom generated by the absence of strict convexity of the corresponding Lagrangian with respect to the generalized velocities. Second, the function

$$\boldsymbol{\mu}\cdot\boldsymbol{q}+2\,\boldsymbol{r}\cdot\boldsymbol{n} \tag{3.19}$$

is an integral. Thus in the quaternion description (3.15) there are, in general, always eight integrals, namely (3.16)–(3.19), and isotropy and uniformity can yield two additional integrals.

(iv) *Integrals for the tumbling rigid body equations*

We now specialize to the equations of rigid body motion and consider the force vector $\boldsymbol{n}$ as a parameter. Furthermore, we assume a constitutive law of the particular form (2.5), as in the Kirchhoff kinetic analogy. In the reduced non-canonical formulation (3.4), the operator (3.3) has a two-dimensional null-space spanned by the vectors

$$\nabla C_1=\begin{pmatrix} \mathsf{n} \\ \mathsf{m} \end{pmatrix},\quad \nabla C_2=\begin{pmatrix} \mathbf{0} \\ \mathsf{n} \end{pmatrix}.$$

The associated Casimirs $C_1=\mathsf{m}\cdot\mathsf{n}$ and $C_2=\frac{1}{2}|\mathsf{n}|^2$ are the trivial integrals of (3.4), independent of the Hamiltonian. In the non-canonical and the Euler angle representation, $C_1$ and $C_2$ constitute all known integrals of the general system. In the quaternion formulation, the additional integral (3.18) remains (while the integral (3.19) does not persist because it involves the variable $\boldsymbol{r}$). In addition, for a symmetric tumbling rigid body (or Lagrange top), the moment $m_3$ is an integral. The free rigid body has the integral $|\mathsf{m}|$. And finally, the Hamiltonian is an integral, provided that the system is autonomous (as is typically the case).

(v) *Completely integrable systems*

Finally, we describe the various integrable cases that arise in the above Hamiltonian formulations. One needs to verify the appropriate count on integrals in involution to obtain complete integrability (see § 2 *c*). These calculations may be cumbersome, but can be efficiently implemented with a symbolic manipulation package (such as Mathematica). For the non-canonical Hamiltonian system (3.11), there always exist two Casimirs and four integrals, three of which are in involution. Isotropy implies the integral $m_3$, which is involution with all the others. Finally, uniformity implies that the Hamiltonian is an integral, yielding a total of two Casimirs and five integrals in involution. Thus a uniform and isotropic rod is integrable. A parametrization in terms of quaternions (respectively Euler angles) always yields eight (six) integrals,

Table 1. *Numbers of integrals for Hamiltonian formulations describing elastic rods*
(The table indicates both the numbers of integrals that always exist and, in parentheses, the numbers that exist in the presence of the additional symmetries of isotropy and uniformity. The cases in parentheses all correspond to complete integrability.)

| | non-canonical | Euler angles | quaternions |
|---|---|---|---|
| no. of Casimirs | 2 | 0 | 0 |
| no. of integrals | 4 (6) | 6 (8) | 8 (10) |
| no. in involution | 3 (5) | 4 (6) | 5 (7) |

Table 2. *Numbers of integrals for Hamiltonian formulations describing a tumbling body*
(The table indicates both the numbers of integrals that always exist and, in parentheses, the numbers that exist for the case of a Lagrange top. The cases in parentheses all correspond to complete integrability.)

| | non-canonical | Euler angles | quaternions |
|---|---|---|---|
| no. of Casimirs | 2 | 0 | 0 |
| no. of integrals | 0 (2) | 2 (4) | 3 (5) |
| no. in involution | 0 (2) | 2 (3) | 2 (4) |

five (four) of which are in involution. Isotropy and uniformity each yield an additional integral, which raises the total number of integrals in involution to seven (six) and implies complete integrability for the system. A similar count arises for the reduced Hamiltonian formulations describing the dynamics of a tumbling rigid body. The results are summarized in tables 1 and 2.

## 4. The last quadrature

In this section we discuss the steps necessary to recover the full set of variables pertinent to rod mechanics, in particular the spatial coordinates of the centreline $\boldsymbol{r}$, when it is assumed that a solution to the reduced problem corresponding to the frame evolution is known (in some sense). The centreline $\boldsymbol{r}$ is certainly always computed via the constraint (1.1), but we demonstrate that the use of various integrals (i.e. invariant quantities) allows the components of $\boldsymbol{r}$ in the fixed frame to be computed directly, without explicitly having to perform the three quadratures implicit in (1.1). We first review the manner in which two of the quadratures can always be eliminated. We then make the observation that the extra information, and extra integral, in the four degree-of-freedom quaternion Hamiltonian formulation of the frame evolution can be exploited to finesse the third and last quadrature.

We also remark that, despite the kinetic analogy, considerable information concerning the shape of the centreline $\boldsymbol{r}$ is lost if the constraint (1.1) is ignored. In particular, the quadratures in (1.1) often seem to average over the potentially very complicated motion of the tangent indicatrix (i.e. the director $\boldsymbol{d}_3$) to yield a rather regular behaviour of the centreline, at least on relatively long length scales. These striking differences are indicated in figures 2, 3 and 4, which depict the evolution

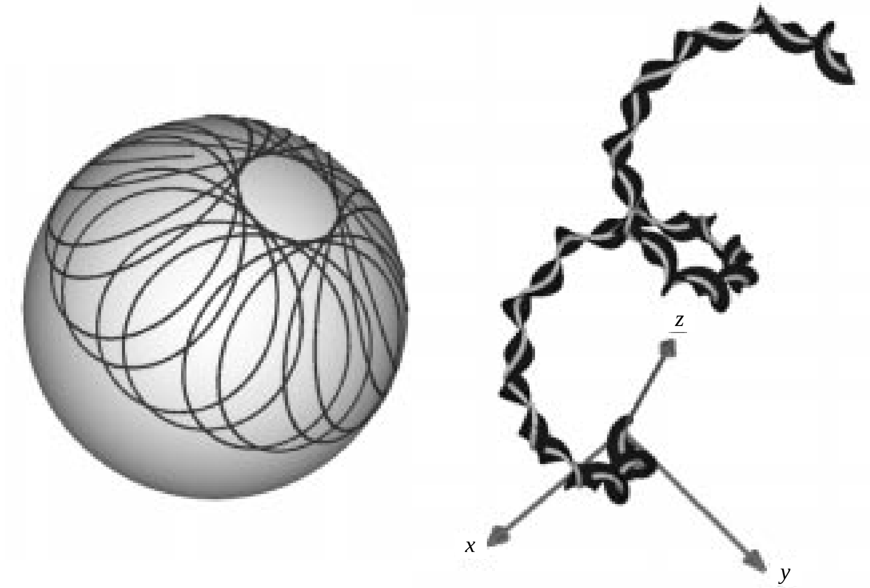


Figure 2. A configuration of an integrable rod and the corresponding tangent indicatrix, tracing the director $\boldsymbol{d}_3$ on the unit sphere. Both representations are highly structured.

of both the tangent indicatrix and the corresponding shape of the centreline $\boldsymbol{r}$ in various integrable and non-integrable cases. In figure 2 a uniform, isotropic rod is shown. The Hamiltonian system is integrable and exhibits a typical 'helix on helix' structure. The corresponding motion of the Lagrange top is indicated by the tangent indicatrix oscillating between two parallel circles on the unit sphere. For a general tumbling rigid body this regular behaviour of the tangent indicatrix is destroyed, as indicated in figures 3 and 4. The tangent indicatrix does not seem to follow any obvious structured pattern. However, the corresponding shape of the rod centreline does, despite a rather irregular local behaviour, average on a longer length scale to a helical structure. A desire to understand this behaviour motivated our consideration of the last quadratures.

### (*a*) *Elimination of two quadratures*

The goal is to compute the components of the centreline in the fixed basis $\{\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\}$, which we assume for convenience to be orthonormal. The vector $\boldsymbol{k}$ is chosen to be parallel to the force vector $\boldsymbol{n}$. Ilyukhin (1979) shows that the $\boldsymbol{i}$ and $\boldsymbol{j}$ components of $\boldsymbol{r}$ are readily computed using the integral (3.16) arising from the balance equations (2.2). Assume that the value of the integral is $\boldsymbol{c}$ and rewrite

$$\boldsymbol{m} + \boldsymbol{r} \times \boldsymbol{n} = \boldsymbol{c},$$

as

$$\boldsymbol{m} + \left(\boldsymbol{r} - \frac{\boldsymbol{n}}{|\boldsymbol{n}|^2} \times \boldsymbol{c}\right) \times \boldsymbol{n} = \frac{\boldsymbol{c} \cdot \boldsymbol{n}}{|\boldsymbol{n}|^2}\boldsymbol{n} = \frac{\boldsymbol{m} \cdot \boldsymbol{n}}{|\boldsymbol{n}|^2}\boldsymbol{n},$$

using the fact that $\boldsymbol{m} \cdot \boldsymbol{n} = \boldsymbol{c} \cdot \boldsymbol{n}$. Without loss of generality we may assume that the fixed coordinate system is translated by a shift $\boldsymbol{r}_0 = \boldsymbol{n}/|\boldsymbol{n}|^2 \times \boldsymbol{c}$, so that

$$\boldsymbol{m} + \boldsymbol{r} \times \boldsymbol{n} = \frac{\boldsymbol{m} \cdot \boldsymbol{n}}{|\boldsymbol{n}|^2}\boldsymbol{n}.$$

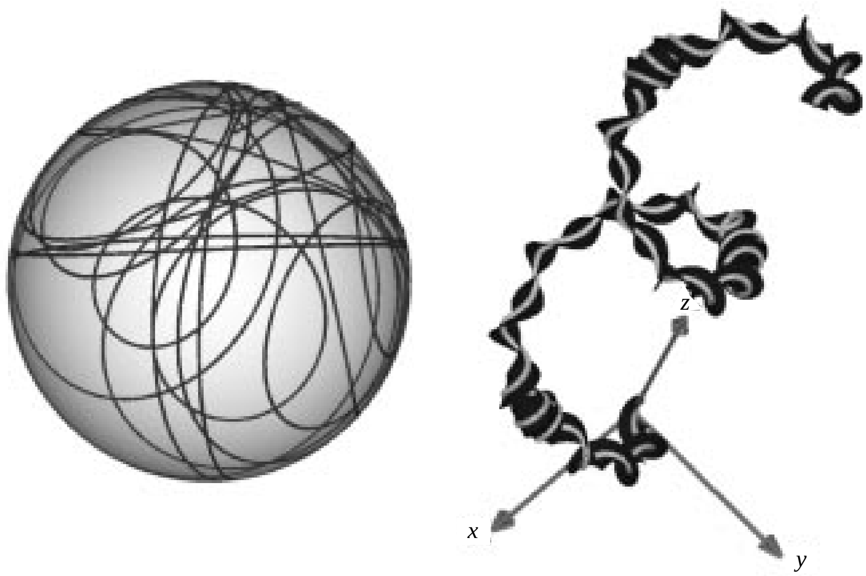


Figure 3. The full configuration of a non-integrable rod exhibits a fairly regular structure, while the tangent indicatrix does not.

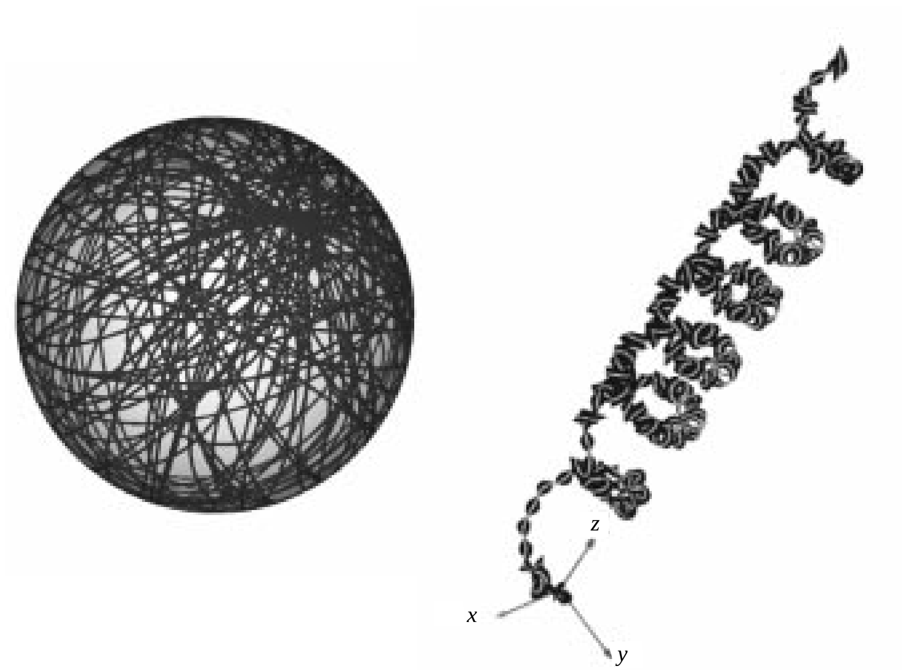


Figure 4. On a relatively long length scale, the rod exhibits a regular averaged structure, which is obscured in the tangent indicatrix.

The components of $\boldsymbol{r}$ with respect to $\boldsymbol{i}$ and $\boldsymbol{j}$ are now readily computed

$$\boldsymbol{r} \cdot \boldsymbol{i} = \frac{\boldsymbol{m} \cdot \boldsymbol{j}}{|\boldsymbol{n}|} = \frac{m \cdot j}{|n|}, \tag{4.1}$$





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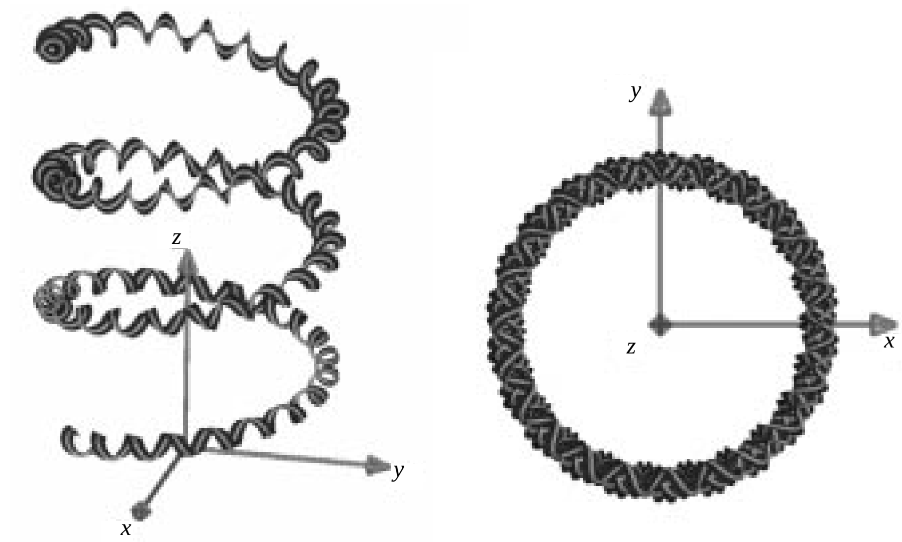


Figure 5. An integrable configuration. The centreline of the rod lies between two concentric circular cylinders (as indicated in the projection).

$$\boldsymbol{r} \cdot \boldsymbol{j} = -\frac{\boldsymbol{m} \cdot \boldsymbol{i}}{|\boldsymbol{n}|} = -\frac{\mathsf{m} \cdot \mathsf{i}}{|\mathsf{n}|}. \tag{4.2}$$

As observed by several authors (cf. Shi & Hearst 1994; Ilyukhin 1979; Langer & Singer 1993), the above argument shows that cylindrical coordinates parallel to the constant force vector $\boldsymbol{n}$ are very special. In particular, the above manipulations allow us to compute the projection of the centreline $\boldsymbol{r}$ into the $\boldsymbol{ij}$-plane. From formulas (4.1) and (4.2) we see that this projection is simply a rotation and scaling of the projection of the moment vector $\boldsymbol{m}$ into the same plane, and two of the quadratures appearing in (1.1) are eliminated.

If the rod is uniform, the Hamiltonian is an integral and we see from (3.10) that $W^*(\mathsf{m}) + \mathsf{m} \cdot \hat{\mathsf{u}}$ is bounded. The coercivity of $W$ is equivalent to the coercivity of $W^*$ (see, for example, Evans 1994, § 3.3.2), i.e. $W^*$ grows super-linearly at infinity, and we conclude that $\mathsf{m}$ is bounded. Thus the projection of $\boldsymbol{r}$ in the $\boldsymbol{ij}$-plane is also bounded; we interpret this boundedness as the centreline of the rod evolving between two concentric cylinders. Figures 5–7 illustrate this conclusion for various integrable and non-integrable, linearly elastic, uniform rod configurations. In each figure we show a skew projection of the rod together with a projection into the $\boldsymbol{ij}$-plane. For the isotropic rod, depicted in figure 5, this conclusion was known, as it follows from the quasi-periodic behaviour of the solution. However, the more general, non-integrable case in figure 6 illustrates that, despite the generally complicated behaviour of the solution, the projection parallel to the $\boldsymbol{n}$-axis is bounded and in fact may resemble that of an isotropic rod.

It is also possible that the radius of the inner cylinder can vanish. In fact this happens exactly if $\boldsymbol{m}(s)$ is parallel to $\boldsymbol{n}(s)$ at some value of $s$. Vanishing of the inner radius is necessary for configurations to exhibit localization. However, there are also non-localized rod configurations for which the radius of the inner cylinder vanishes (cf. figure 7). Again, in this configuration the skew projection appears rather complicated, while the projection along the $\boldsymbol{n}$-axis has an almost periodic behaviour.

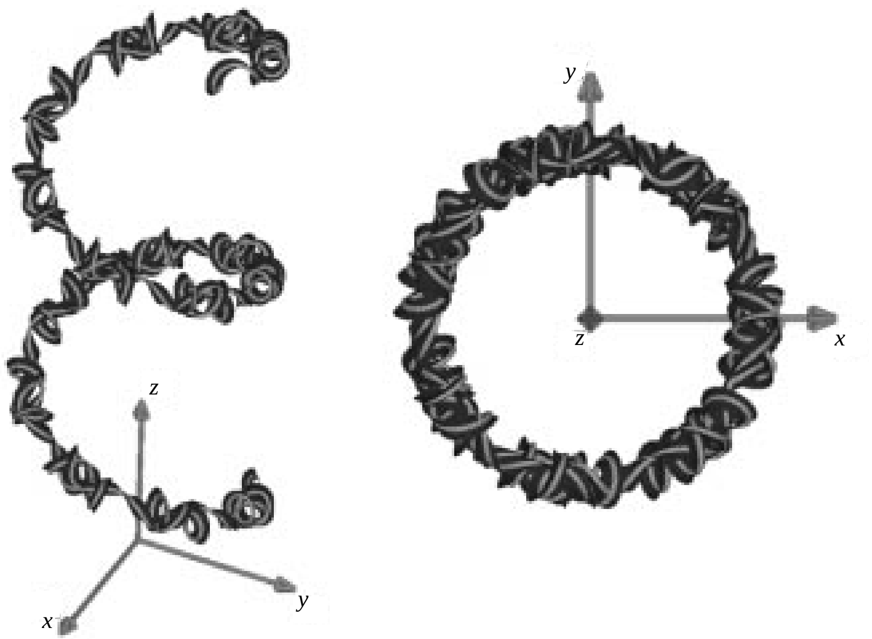


Figure 6. For a non-integrable rod, the overall structure becomes rather complicated. However, the existence of the two bounding cylinders persists.

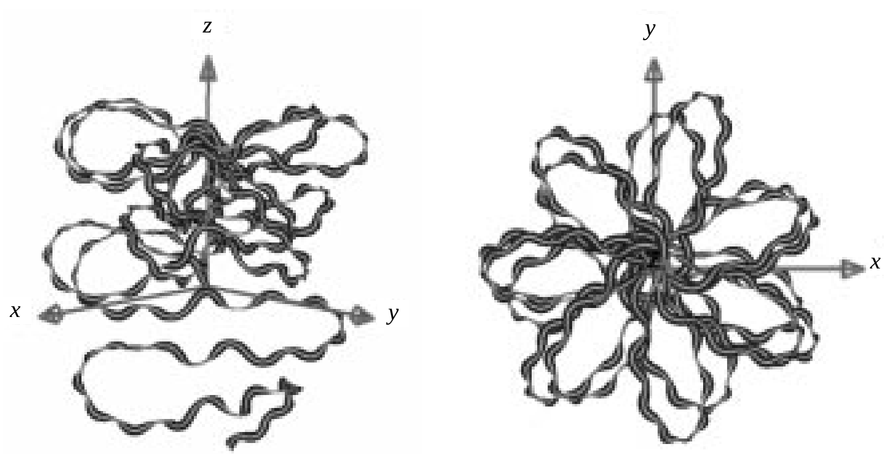


Figure 7. A configuration where the radius of the inner cylinder vanishes. The side view suggests a complicated structure, while the projection seems almost periodic.

### (*b*) *The last quadrature resolved*

It remains to compute the $\boldsymbol{k}$ component of the centreline $\boldsymbol{r}$ via the *last quadrature*

$$\boldsymbol{r}' \cdot \boldsymbol{k} = \boldsymbol{d}_3 \cdot \boldsymbol{k} = \frac{n_3}{|n|}.$$

When an integrable rod is of interest it is possible to explicitly carry out this quadrature. Our observation is that if a solution to the reduced frame evolution is known in terms of quaternions, then, whether or not the rod is integrable, the last quadrature can be performed explicitly.





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Consider the specific representation of the Hamiltonian system (3.15) in terms of quaternions. It is only here that the integral (3.19) arises. Assuming that the value of this integral is $\boldsymbol{\mu} \cdot \boldsymbol{q} + 2\boldsymbol{r} \cdot \boldsymbol{n} = c$, the last quadrature is

$$\boldsymbol{r} \cdot \boldsymbol{k} = \frac{1}{2|\boldsymbol{n}|}(\boldsymbol{\mu} \cdot \boldsymbol{q} - c). \tag{4.3}$$

The value of $c$ merely represents a shift in origin of $\boldsymbol{r}$ parallel to $\boldsymbol{n}$.

We may therefore conclude that the extra degree of freedom in a quaternion Hamiltonian description of the equilibria of elastic rods contains additional information that provides a strengthened kinetic analogy. The relations (4.1), (4.2) and (4.3) allow a complete reduction of two-point boundary value problems for rods to a two-point boundary value problem for the lower-dimensional reduced quaternion Hamiltonian description of the frame evolution. This fact has been exploited for numerical simulations by Warner (1997), and he observes an increase in performance for the reduced problem by approximately a factor of three.

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